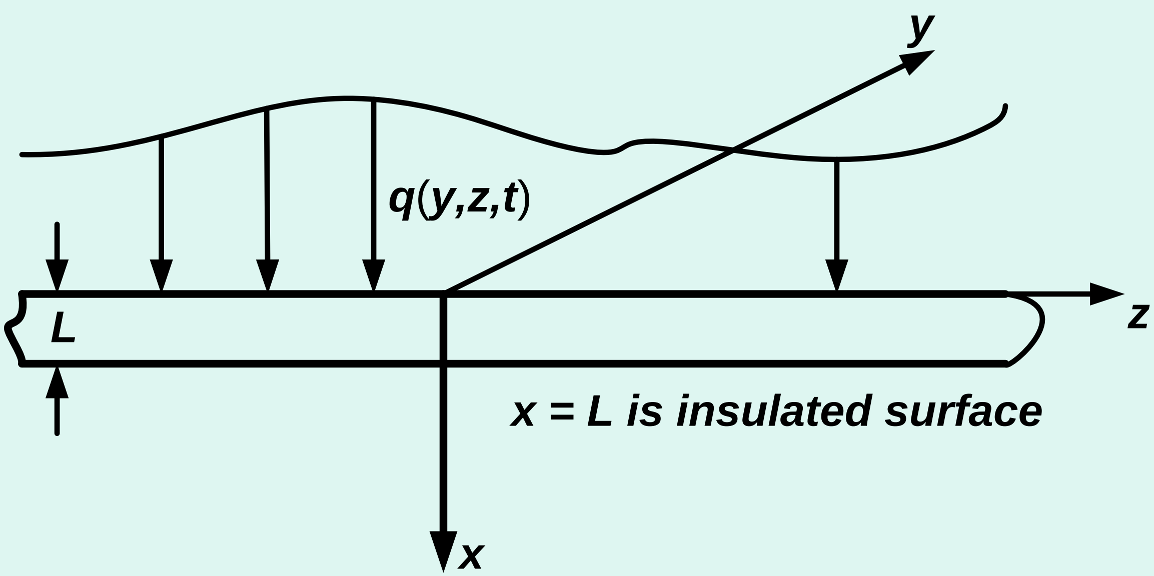
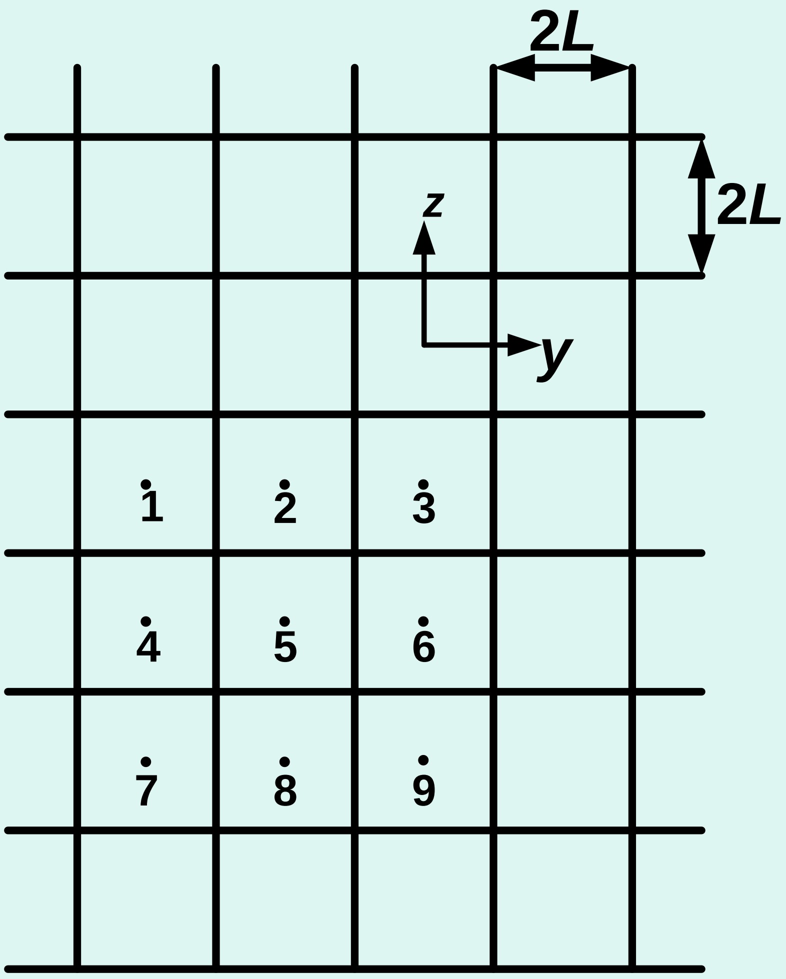
Filter Inverse Heat Conduction 3D Solution for Large Plates

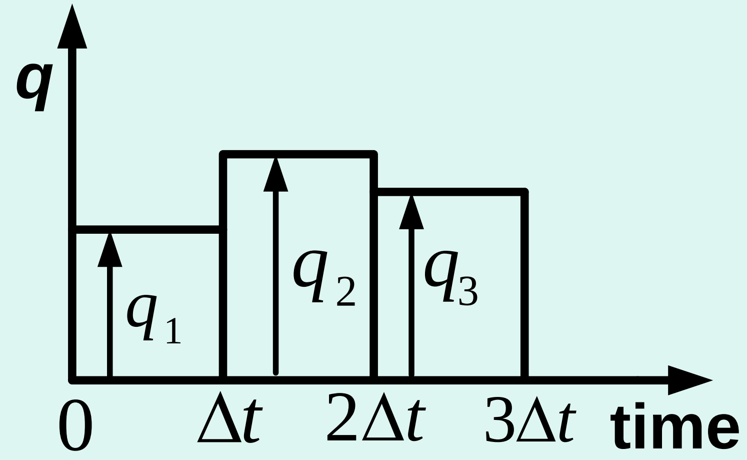
James V. Beck
Prof. Em., Dept. of Mech. Eng.
Michigan State Univ.
East Lansing, MI 48864
beck@msu.edu



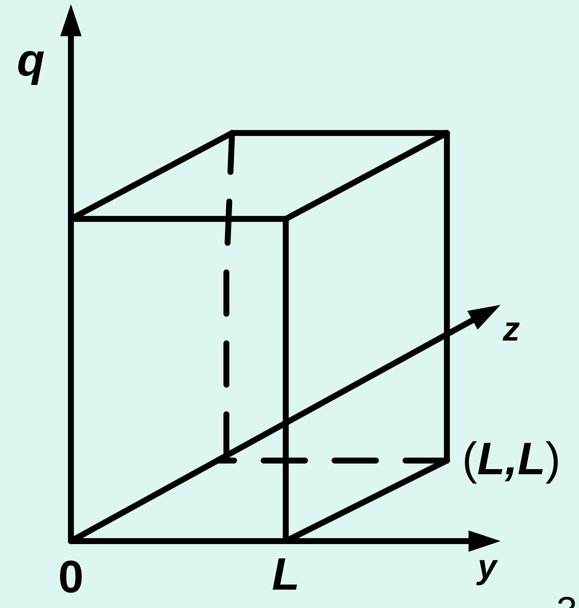
The heat flux *q*(*y,z,t*) is approximated in space by a checkerboard pattern with uniform elements in space and step changes in time.



2L
2L
z
y
1
2
3
4
5
6
7
8
9



q
q_1
q_2
q_3
0
Δt
$2\Delta t$
$3\Delta t$
time



q
z
(L,L)
0
L
y

MATHEMATICAL MODEL FOR BUILDING BLOCK

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0<x<L, \quad 0<y<\infty, \quad 0<z<\infty, \ t>0$$

$$-k\frac{\partial T}{\partial x}(0,y,z,t)=\begin{cases} q_0, \ 0<y<L, \ 0<z<L \\ 0, \ \text{otherwise} \end{cases}, \quad \frac{\partial T}{\partial x}(L,y,z,t)=0$$

$$\frac{\partial T}{\partial y}(x,0,z,t)=0, \ T(x,\infty,z,t) \text{ is finite}$$

$$\frac{\partial T}{\partial z}(x,y,0,t)=0, \ T(x,y,\infty,t) \text{ is finite}$$

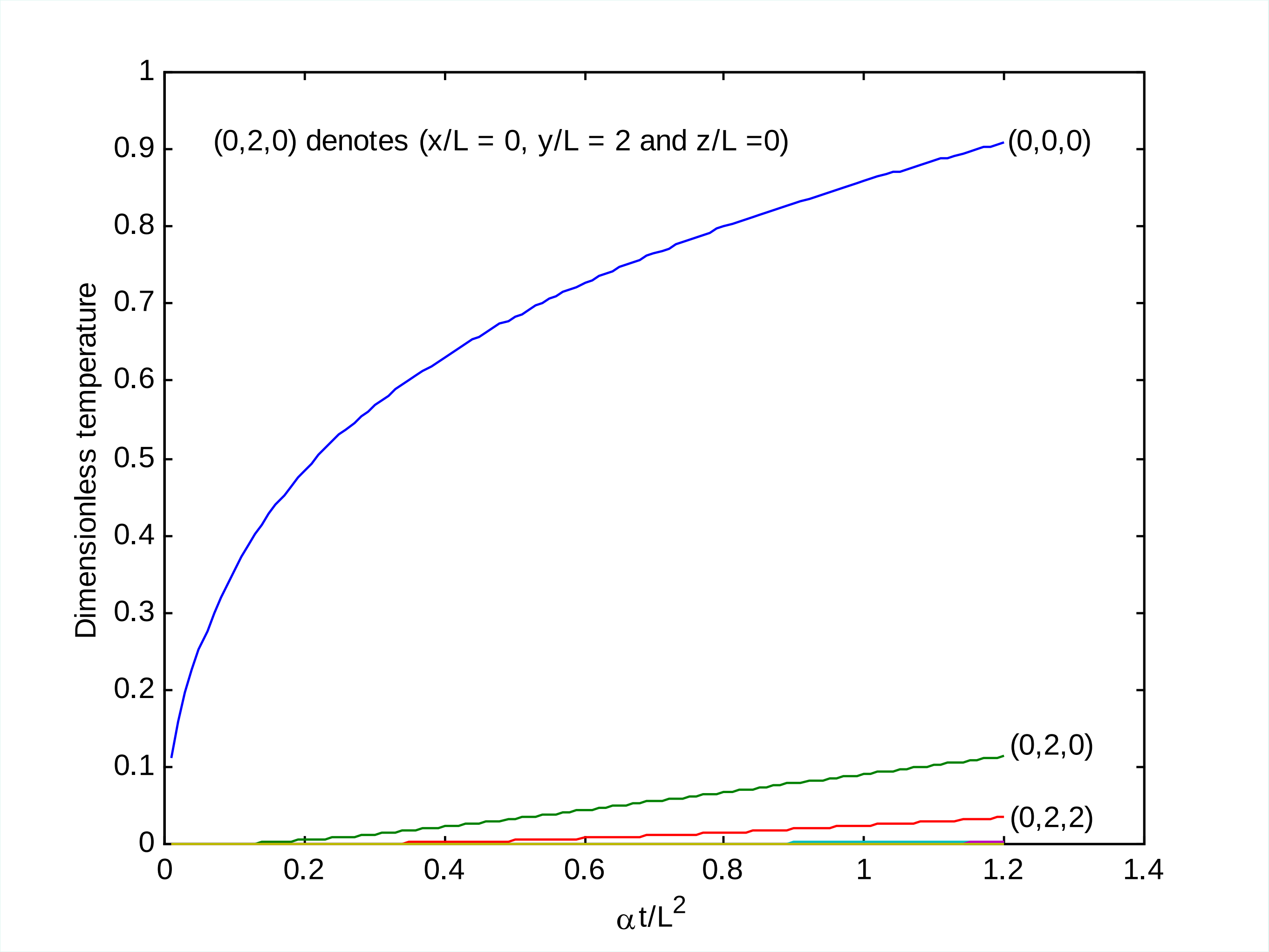
$$T(x,y,z,0)=0$$

DIMENSIONLESS FORM:

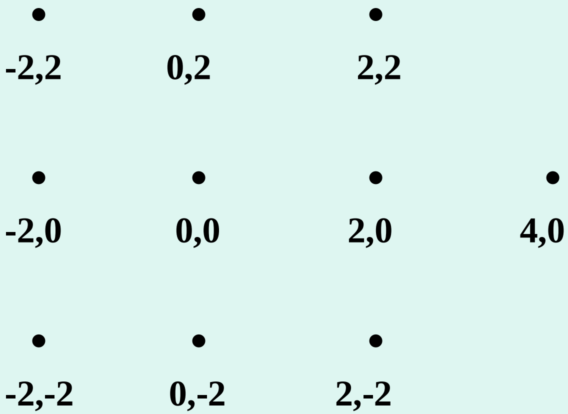
$$\tilde{T}\equiv\frac{T(x,y,z,t)}{\frac{q_0 L}{k}}=\tilde{T}\left(\tilde{x},\tilde{y},\tilde{z},\tilde{t}\right)$$

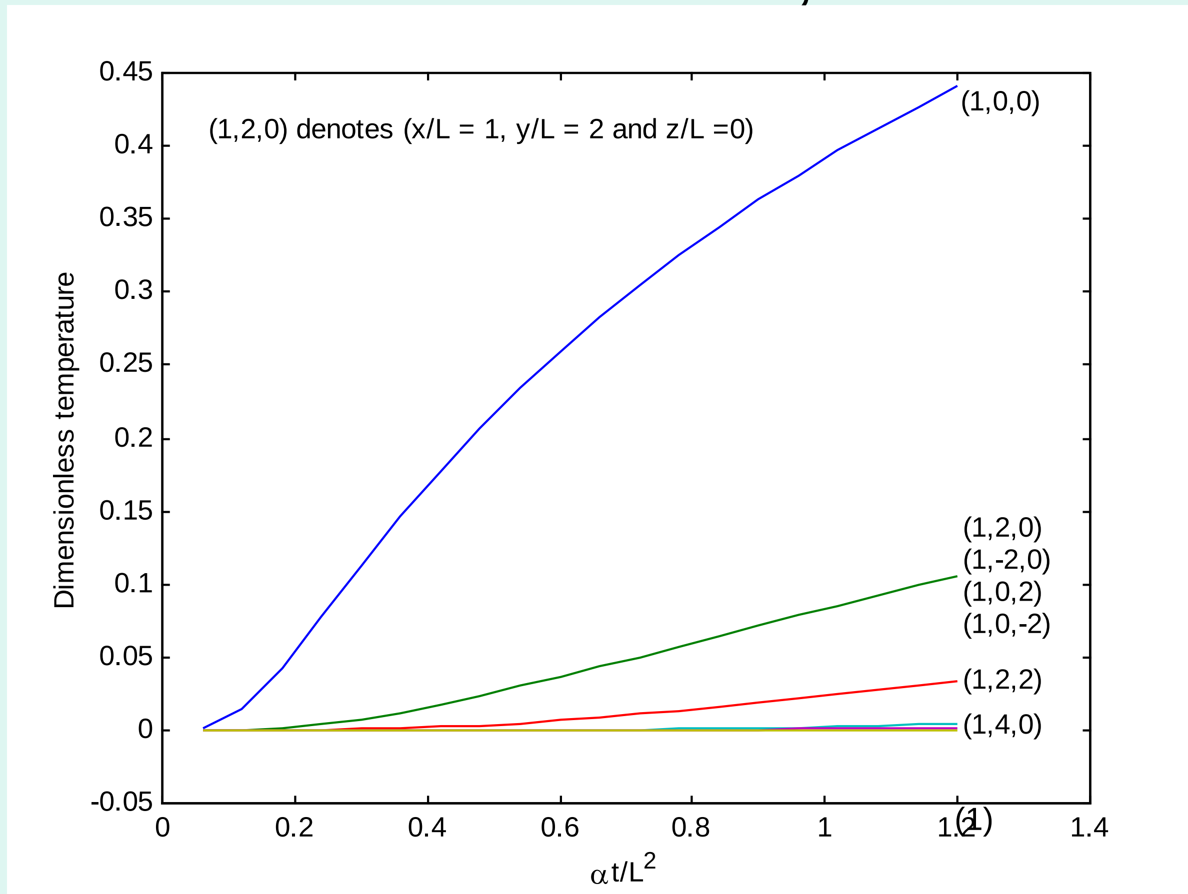
$$\tilde{x}=\frac{x}{L}, \ \tilde{y}=\frac{y}{L}, \ \tilde{z}=\frac{z}{L}, \ \tilde{t}=\frac{\alpha t}{L^2}$$

HEATED SURFACE TEMPERATURES



DIMENSIONLESS TEMPERATURE FOR BUILDING BLOCK, *x = L*





$\tilde{t} = \dfrac{\alpha t}{L^2}$	(1, 0, 0)	(1, 2, 0)	(1,2, 2)	(1, 4, 0)
0.060	0.000782	0.000001	0.000000	0.000000
0.120	0.014205	0.000168	0.000002	0.000000
0.180	0.042480	0.001231	0.000045	0.000000
0.240	0.076840	0.003646	0.000218	0.000000

TEMPERATURE MEASURED AT $x = L$

THESE T'S ARE DAMPED AND LAGGED

COMPARED WITH THOSE AT (0,0,0)

LOCAL IN TIME AND SPACE MEASUREMENTS

ARE IMPORTANT

HEAT FLUX AT GIVEN LOCATION AT $x = L$ MOST

INFLUENCED IN SPACE BY HEATING DIRECTLY

ABOVE BUT IN TIME LATER (LAG)

RECALL

$$\tilde{T} \equiv \frac{T(x,y,z,t)}{\frac{q_0 L}{k}} = \tilde{T}\left(\tilde{x},\tilde{y},\tilde{z},\tilde{t}\right), \quad T(x,y,z,t) = \frac{q_0 L}{k}\tilde{T}\left(\tilde{x},\tilde{y},\tilde{z},\tilde{t}\right)$$

SENSITIVITY COEFFICIENTS

$$\frac{\partial T}{\partial q_0}(i) = Z(i) = \frac{L}{k}\tilde{T}(i)$$

Same as temperature rise at time *i* for *q* = 1

Now use FUNCTION SPECIFICATION METHOD

FOR A GIVEN TIME STEP TEMPORARILY HOLD *q* CONSTANT IN TIME.

CALCULATED TEMPERATURE VECTOR: $\mathbf{T} = \mathbf{T}\big|_{\mathbf{q=0}} + \mathbf{Z}\,\mathbf{q}$

MEASURED TEMPERATURE VECTOR: $\mathbf{Y}$

MATRIX SUM OF SQUARES FUNCTION:

$$S = (\mathbf{Y} - \mathbf{T}\big|_{\mathbf{q=0}} - \mathbf{Z}\,\mathbf{q})^T (\mathbf{Y} - \mathbf{T}\big|_{\mathbf{q=0}} - \mathbf{Z}\,\mathbf{q})$$

ESTIMATED HEAT FLUX VECTOR:

$$\hat{\mathbf{q}} = (\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T (\mathbf{Y} - \mathbf{T}\big|_{\mathbf{q=0}})$$

THIS EQ. CAN BE FOR SCALAR (1D) OR VECTOR

FOR 3D PROBLEM BUT 2D SURFACE

FOR 1D PROBLEM WITH r = 4 FUTURE STEPS:

$$\mathbf{Z}^{[4\mathbf{x}1]} = \begin{bmatrix} Z(1) \\ Z(2) \\ Z(3) \\ Z(4) \end{bmatrix}^{[4\mathbf{x}1]}$$

$$\hat{\mathbf{q}}^{[1\mathbf{x}1]} = (\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T(\mathbf{Y} - \mathbf{T}\big|_{\mathbf{q}=\mathbf{0}})$$

$$(\mathbf{Z}^T\mathbf{Z})^{[1\mathbf{x}1]}$$

$$\mathbf{F} = \left((\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T\right)^{[p\ \mathbf{x}\ pr]}$$

FOR *p* SPATIAL VALUES OF *q & r* FUTURE STEPS

$$\mathbf{Z}^{[rp\mathbf{x}p]} = \begin{bmatrix} \mathbf{Z}(1) \\ \mathbf{Z}(2) \\ \vdots \\ \mathbf{Z}(r) \end{bmatrix}, \ \mathbf{Z}(1)^{[p\mathbf{x}p]} = \begin{bmatrix} Z_{11}(1) & Z_{12}(1) & \cdots & Z_{1p}(1) \\ Z_{21}(1) & Z_{22}(1) & \cdots & Z_{1p}(1) \\ \vdots & \vdots & \ddots & \vdots \\ Z_{p1}(1) & Z_{p2}(1) & \cdots & Z_{pp}(1) \end{bmatrix}$$

CONSIDER AN ELEMENT WITH 8 SURROUNDING

1	2	3	-2,2	0,2	2,2	
4	5	6	-2,0	0,0	2,0	4,0
7	8	9	-2,-2	0,-2	2,-2	

	a_1(i)	a_2(i)	a_3(i)	
$\alpha\Delta t / L^2$	(1, 0, 0)	(1, 2, 0)	(1,2, 2)	(1, 4, 0)
0.060	0.000782	0.000001	0.000000	0.000000
0.120	0.014205	0.000168	0.000002	0.000000
0.180	0.042480	0.001231	0.000045	0.000000
0.240	0.076840	0.003646	0.000218	0.000000

$$
\mathbf{Z}(i) = \begin{bmatrix}
a_1(i) & a_2(i) & 0 & a_2(i) & a_3(i) & 0 & 0 & 0 & 0 \\
a_2(i) & a_{1(i)} & a_2(i) & a_3(i) & a_2(i) & a_3(i) & 0 & 0 & 0 \\
0 & a_2(i) & a_1(i) & 0 & a_3(i) & a_2(i) & 0 & 0 & 0 \\
a_2(i) & a_3(i) & 0 & a_1(i) & a_2(i) & 0 & a_2(i) & a_3(i) & 0 \\
a_3(i) & a_2(i) & a_3(i) & a_2(i) & a_1(i) & a_2(i) & a_3(i) & a_2(i) & a_3(i) \\
0 & a_3(i) & a_2(i) & 0 & a_2(i) & a_1(i) & 0 & a_3(i) & a_2(i) \\
0 & 0 & 0 & a_2(i) & a_3(i) & 0 & a_1(i) & a_2(i) & 0 \\
0 & 0 & 0 & a_3(i) & a_2(i) & a_3(i) & a_2(i) & a_1(i) & a_2(i) \\
0 & 0 & 0 & 0 & a_3(i) & a_2(i) & 0 & a_2(i) & a_1(i)
\end{bmatrix}
$$

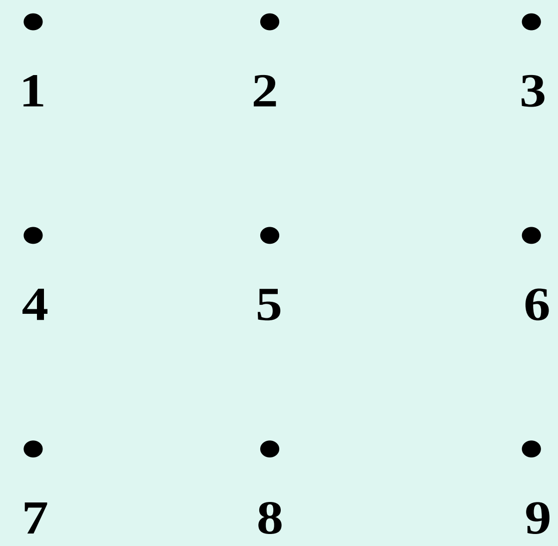
NOTICE THE SYMMETRY AND PATTERN OF THE 5TH ROW AND COLUMN: 1 2 3 2 3

SAME PATTERN IS NOTED IN: $\mathbf{F} = \left((\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T \right)^{[p \ \mathbf{x} \ pr]}$

$$\mathbf{F} = \left((\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T \right)^{[p \, \mathbf{x} \, pr]} =$$

$$\left[\mathbf{F}(1) \quad \mathbf{F}(2) \quad \mathbf{F}(3) \quad \mathbf{F}(4) \right]$$

$$\mathbf{F}^{\nearrow}(1) = \begin{matrix} 0.000566 & -0.000027 & 0.000518 & -0.000027 & 0.000001 \\ -0.008377 & 0.000566 & -0.000027 & 0.000519 & -0.000027 \\ 0.000566 & -0.008350 & 0.000001 & -0.000027 & 0.000518 \\ -0.008377 & 0.000519 & -0.008350 & 0.000566 & -0.000027 \\ 0.100949 & -0.008377 & 0.000566 & -0.008377 & 0.000566 \end{matrix}$$



i	f_1(i)	$f_2(i)$	$f_3(i)$
1	0.100949	-0.008377	0.000566
2	1.826514	-0.132168	0.007273
3	5.430662	-0.300903	0.010760
4	9.762114	-0.361938	0.006362

SCALAR EQUATION FOR THE "CENTER" POINT, #5

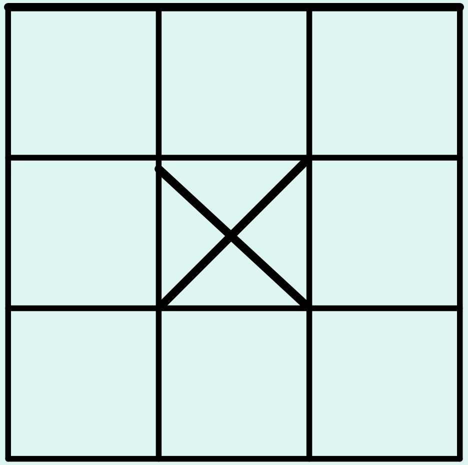
$$\hat{q}_5(M) = \sum_{i=1}^{r} \left[\begin{array}{l} f_1(i)\left(Y_5(M+i-1) - T_5(M+i-1)\Big|_{q_M=\ldots=0}\right) \\ +S_2(M,i) + S_3(M,i) \end{array} \right]$$

$$S_2(M,i) = f_2(i) \sum_{j=2,4,6,8} \left(Y_j(M+i-1) - T_j(M+i-1)\Big|_{q_M=\ldots=0}\right)$$

$$S_3(M,i) = f_3(i) \sum_{j=1,3,5,7} \left(Y_j(M+i-1) - T_j(M+i-1)\Big|_{q_M=\ldots=0}\right)$$

CONSIDER 1ST TIME STEP & INITIAL T = 0

$$\hat{q}_5(1) = \sum_{i=1}^{r} \left[f_1(i) Y_5(i) + f_2(i) \sum_{j=2,4,6,8} Y_j(i) + f_3(i) \sum_{j=1,3,5,7} Y_j(i) \right]$$



ONLY CENTER POINT HEATED

USING *r* = 4, DIM. TIME STEP: 0.06

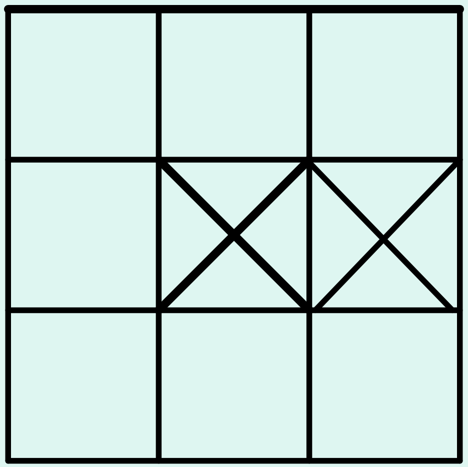
DIMENSIONLESS HEAT FLUX AT

CENTER POINT IS 0.999998

VERY ACCURATE RESULT. CONSIDER ADDITIVE, INDEPENDENT, NORMAL ERRORS OF 5% OF TEMP. RISE OF 0.08 AT TIME = 0.24 AT *x* = *L*, 0.004

SOME RESULTS: 1.016, 0.989, 0.972, 0.995, 0.955

EXAMPLE WITH TWO HEATED ELEMENTS



USING SAME 9 SURROUNDING

LOCATIONS AND 4 TIMES:

q = 0.999995

WITH SAME MAGNITUDE RANDOM ERRORS AS BEFORE:

q = 0.930, 1.029, 0.968, 1.008

FINAL POINTS

1. It is possible to estimate locally in time and space the surface heat flux, for a large plate and small time steps.

2. The solution has two parts, local in time and space and global in space/time. Compute *q* and use in

$$\mathbf{T}(M) = \mathbf{T}(\mathrm{M})\Big|_{\mathbf{q}=0} + \mathbf{Z}(1)\mathbf{q}(M)$$ for one time step

Local Global Local